An Improvement to an Approximation Algorithm for Max k-CSP(d)
REU Report

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based on joint work with
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1 Introduction

In this technical report, we present a slight improvement to an algorithm for the Max k-CSP(d) problem, the maximum constraint satisfaction problem with at most k variables in a clause and a domain of size d. The base of the algorithm is that of Makarychev and Makarychev [4]. As noted in their paper, their algorithm is efficient in the domain where $k \geq \Omega(\log d)$. Here we present an approach that extends theirs and provides the same approximation ratio, $\Omega(kd/d^k)$, for all instances of the problem. The previous best approximation result for the domain where $k < O(\log d)$ was $\Omega(d/d^k)$, also presented in [4].

In this report, we have also tried to improve the approximation constants in [4], as suggested by the authors in their paper. We present improvements in both the uniform and arbitrary rounding schemes, which result in the approximation algorithm having a constant of 1/8. It is the hope of the authors to further optimize this algorithm and get an approximation ratio of $\alpha kd/d^k$ where $\alpha \to 1$ as $k, d \to \infty$.

There have been conclusive studies on some smaller families of CSPs. As will be clear from the definitions, problems such as 3-SAT and Max Cut are both special instances of the boolean (i.e. $d = 2$) Max k-CSP problem. Both problems have known algorithms and matching hardness of approximation results, although in the latter case it is only under the assumption that the Unique Games Conjecture (UGC) is true. In a more general sense, the boolean Max k-CSP is asymptotically closed: the current best algorithms meet the hardness of approximation results, $\Theta(k/2^k)$, under $P \neq NP$ assumptions.

It has been shown that the non-boolean case is hard to approximate within $O(d^2/d^k)$ [1], [3]. Moreover, for the cases where $k = (d^t - 1)/(d - 1)$ the hardness result is $O(kd/d^k)$. This was later improved to the case where $k \geq d$ by Håstad (see [4] for his proof). It was recently shown by Chan [2] that hardness of $O(kd/d^k)$ holds for $k \geq d$ only under the assumptions that $P \neq NP$. These results provide some evidence to believe that the general Max k-CSP(d) problem is hard to approximate within a ratio better than $O(kd/d^k)$, although at the time of writing there is no conclusive proof. If this result was shown to be true, then the algorithm presented on this report would be asymptotically optimal.

Overview In section 2, we define the problem and state some lemmas that will be essential for the main proofs. In section 3, we describe a rounding scheme for the case when the SDP solution vectors have the same length. This scheme has two components, each more effective than the other on a specific domain. In section 4, we present two rounding schemes for rounding arbitrary solutions. The first is a reduction to the previous case. The second rounding scheme itself has two components, the latter of which is the novelty with respect to [4]. In section 5, we present the main result of this report: an $\Omega(kd/d^k)$ approximation algorithm for the Max k-CSP(d) problem.

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2 Preliminaries

The Constraint Satisfaction Problem (CSP) encompasses a large family of problems in combinatorial optimization. Some of these problems include notorious NP-Hard problems such as 3$-$SAT or Max Cut. These problems motivate the study of approximation algorithms for the more general class of CSPs. It is known that CSPs are NP-Hard to optimize over. Therefore, the most interesting results in this field are concerned with approximation algorithms and matching hardness of approximation proofs.

An instance of a $k-$CSP(d) is a set of clauses (also known as constraints) C and a set of indices X such that every clause contains at most k indices that take a value in the domain $[0, 1, ..., d-1]$ of size d. An approximation preserving reduction due to Trevisan [6] allows us to assume that all clauses are conjunctions of the form $(x_{u_1} = i_1 \wedge ... \wedge x_{u_k} = i_k)$, where $x_u \in X$. We say that an assignment x^* satisfies a clause $c \in C$ if for every pair $(u, i) \in C$ we have that $x^*_u = i$. The Max k-CSP(d) problem is concerned with finding an assignment of the indices that maximizes the fraction of satisfied clauses.

A result by Raghavendra [5] shows that under the Unique Games Conjecture (UGC), the best approximation algorithms for CSPs come from simple Semi-Definite Programs (SDP). This motivates the use of SDPs and rounding schemes to produce approximation algorithms for the CSP problems. Consider the following semi-definite program, where we have a vector variable u_i for every pair (u, i) of indices and domain values. Similarly, for every clause $c \in C$ we have a vector variable z_c.

$$\begin{aligned}
\text{maximize} \quad & \sum_{c \in C} ||z_c||^2 & \\
\text{subject to} \quad & \sum_{i \in [d]} ||u_i||^2 = 1 & \forall u \in X \\
& \langle u_i, z_c \rangle = ||z_c||^2 & \forall (u, i) \in C \\
& \langle u_j, z_c \rangle = 0 & \forall (u, j) \notin C \\
& \langle u_i, u_j \rangle = 0 & \forall u \in X, i \neq j, i, j \in [d]
\end{aligned}$$

Let OPT(I) be the optimal solution for a given instance I and x^* an assignment that achieves it. It is not difficult to construct a solution for the semi definite program above. Consider the following SDP(I) solution

$$u_i = \begin{cases} e & \text{if } (u, i) \in c \\ 0 & \text{if } (u, i) \notin c \end{cases}$$

$$z_c = \begin{cases} e & \text{if } c \text{ is satisfied} \\ 0 & \text{if } c \text{ not satisfied} \end{cases}$$

This solution satisfies all the SDP constraints and therefore achieves a value SDP(I) $\geq$ OPT(I).

Definition. We say that an SDP solution is uniform if $||u_i||^2 \leq \frac{1}{d}$ for all $u \in X, i \in [d]$.

Definition. For any clause $c \in C$, let $\text{supp}(c) = \{x \in X | x \in c\}$. In other words, the support of a clause is the set of indices in it.

Definition. Let ξ be a standard Gaussian random variable with mean 0 and variance 1. Throughout this report let

$$\Phi(t) = \Pr(|\xi| \leq t)$$
$$\overline{\Phi(t)} = 1 - \Phi(t) = \Pr(|\xi| > t).$$

We will use the following results, as shown in [4].

Lemma 1. *(Makarychev and Makarychev, [4]) For every $t > 0, \beta \in (0,1]$ we have that*

$$\overline{\Phi(\beta t)} \leq \overline{\Phi(t)}^{\beta^2}.$$

Lemma 2. *(Šidák, [7]) Let $\xi_1, ..., \xi_r$ be jointly distributed Gaussian random variables with mean 0 and arbitrary covariance. Then for any positive t_i*

$$\Pr(|\xi_1| \leq t_1, ..., |\xi_r| \leq t_r) \geq \prod_{i=1}^{r} \Pr(|\xi_i| \leq t_i).$$

3 Rounding uniform SDP solutions

Lemma 3. *There is a randomized polynomial-time algorithm that given an instance I of the Max $k-CSP(d)$ problem (for sufficiently large d) and a uniform SDP solution, outputs an assignment x such that for every clause $c \in C$*

$$\Pr(c \text{ is satisfied by } x) \geq \frac{c_k ||z_c||^2 kd}{d^k}$$

where c_k is a constant that depends on k such that $\lim_{k\to\infty} c_k = 1$.

Proof. Consider the following rounding algorithm:
Input: an instance of Max k-CSP(d) and a uniform SDP solution.
Output: an assignment x.

- Choose a random Gaussian vector g such that every component has mean 0 and variance 1, and all components are independent.
- For every $u \in X$, let $x'_u = \arg\max_i |\langle g, u_i\rangle|$.
- For every $u \in X$, choose x'' uniformly at random from $[d]$.

- With probability $\frac{k}{e^k}$ use assignment x', otherwise use assignment x''.

For every clause $c \in C$ we will estimate the probability that x', x'' satisfy c and then show that the expected value is the one stated above. It is clear that assignment x'' will satisfy c with probability $\frac{1}{d^{|c|}}$. We now show that assignment x' satisfies c with probability $d^{-|c|(1/2+1/(2(1+\beta/2)))}2^{-|c|}$, where $\beta > 0$ is a constant.

Claim 1. *Suppose $c \in C$ is a clause such that $||z_c||^2 \geq (1+\beta)/(|c|d)$ for some $\beta > 0$. Then the probability that the assignment x' satisfies c is at least $d^{-|c|(1/2+1/(2(1+\beta/2)))}2^{-|c|}$.*

Proof. Let $s = |c|$. Assume without loss of generality that for all $u \in \text{supp}(c), (u,1) \in c$. From the SDP constraints we have that $||z_c||^2 = \langle z_c, u_i \rangle \leq ||z_c|| \cdot ||u_i|| \leq ||z_c||/\sqrt{d}$. Therefore we get that $||z_c||^2 \leq 1/d$. Moreover, we also have that $s \geq (1+\beta)$ since $||z_c||^2 \geq (1+\beta)/(sd)$ by assumption.

For every $u \in \text{supp}(c)$, let $u_1^{\perp} = u_1 - z_c$. Let $\gamma_c = \langle g, z_c \rangle$, $\gamma_{u,1} = \langle g, u_1^{\perp} \rangle, \gamma_{u,i} = \langle g, u_i \rangle$ for $i \geq 2$. All variables $\gamma_{u,i}, \gamma_{u,1}$ are jointly Gaussian random variables. Using the fact that for vectors v and w $\mathbb{E}[\langle g, v \rangle \cdot \langle g, w \rangle] = \langle v, w \rangle$ we get that

$$\begin{aligned}
\mathbb{E}[\gamma_c \gamma_{u,1}] &= \langle z_c, u_1^{\perp} \rangle = \langle z_c, u_1 - z_c \rangle = 0 \\
\mathbb{E}[\gamma_c \gamma_{u,i}] &= \langle z_c, u_i \rangle = 0
\end{aligned}$$

Therefore, all $\gamma_{u,i}$ are independent of γ_c. Let $M = \overline{\Phi}^{-1}(1/d^{s/(1+\beta/2)})/\sqrt{sd/(1+\beta)}$. We write the probability that x' satisfies c.

$$\begin{aligned}
\Pr(x' \text{satisfies } c) &= \Pr(\arg\max_i |\langle g, u_i \rangle| = 1 \forall u \in \text{supp}(c)) \\
&= \Pr(|\langle g, u_1 \rangle| > |\langle g, u_i \rangle| \forall i \geq 2, u \in \text{supp}(c)) \\
&= \Pr(|\gamma_{u,1} + \gamma_c| > |\gamma_{u,i}| \forall i \geq 2, u \in \text{supp}(c)) \\
&\geq \Pr(|\gamma_{u,1}| \leq \delta M, |\gamma_{u,i}| \leq (1-\delta)M \\
&\quad \forall i \geq 2, u \in \text{supp}(c) | |\gamma_c| > M) \cdot \Pr(|\gamma_c| > M).
\end{aligned}$$

where $\delta > 0$ is a constant such that $(1-\delta)^2 = \frac{1+\beta/2}{1+\beta}$. Since all variables $\gamma_{u,i}$ are independent from γ_c, we get that

$$\Pr(|\gamma_{u,1}| \leq \delta M), |\gamma_{u,i}| \leq (1-\delta)M \forall i \geq 2, u \in \text{supp}(c)) \cdot \Pr(|\gamma_c| > M)$$

By Šidák's Theorem, we can bound the first term.

$$\Pr(x' \text{satisfies } c) \geq \prod_{u \in \text{supp(c)}} \left(\Pr(|\gamma_{u,1}| \leq \delta M) \prod_{i \geq 2} \Pr(|\gamma_{u,i}| \leq (1-\delta)M) \right) \cdot \Pr(|\gamma_c| > M)$$

In order to estimate these probabilities we need to compute the variances of the vectors $\gamma_{u,i}$. We use the fact that $\mathrm{Var}[\langle g, v \rangle] = ||v||^2$ for any vector v. Therefore,

$$\mathrm{Var}[\gamma_{u,1}] = ||u_1^{\perp}||^2 = ||u_1||^2 + ||z_c||^2 - 2\langle u_1, z_c \rangle = ||u_1||^2 - ||z_c||^2 \leq ||u_1||^2 \leq 1/(d)$$
$$\mathrm{Var}[\gamma_{u,i}] = ||u_i||^2 \leq 1/(d)$$

We can now estimate the probabilities above. We will use the facts that $\Phi(t)$ is an increasing function and $\overline{\Phi(\beta t)} \leq \overline{\Phi(t)}^{\beta^2}$, where $\beta \in (0, 1]$. Therefore, we get that for $i \geq 2$.

$$\begin{aligned}
\Pr(|\gamma_{u,i}| \leq (1-\delta)M) &= \Phi((1-\delta)M/(\sqrt{\mathrm{Var}[\gamma_{u,i}]})) \geq \Phi(\sqrt{d}(1-\delta)M) \\
&= 1 - \overline{\Phi}(\sqrt{d}(1-\delta)M) \geq 1 - \overline{\Phi}(\sqrt{sd/(1+\beta)}M)^{(1+\beta)(1-\delta)^2/s} \\
&= 1 - (d^{-s/(1+\beta/2)})^{(1+\beta)(1-\delta)^2/s} = 1 - d^{-1}
\end{aligned}$$

On the other hand, for $i = 1$ we get that

$$\begin{aligned}
\Pr(|\gamma_{u,1}| \leq \delta M) &= \Phi(\delta M/(\sqrt{\mathrm{Var}[\gamma_{u,i}]})) \geq \Phi(\sqrt{d}\delta M) \\
&= 1 - \overline{\Phi}(\sqrt{d}\delta M) \geq 1 - \overline{\Phi}(\sqrt{sd/(1+\beta)}M)^{(1+\beta)\delta^2/s} \\
&= 1 - (d^{-s/(1+\beta/2)})^{(1+\beta)\delta^2/s} = 1 - d^{-\delta^2/(1-\delta)^2}
\end{aligned}$$

Finally,

$$\begin{aligned}
\Pr(|\gamma_c| > M) &\geq \overline{\Phi}(M/\sqrt{\mathrm{Var}[\gamma_c]}) \\
&\geq \overline{\Phi}(M\sqrt{sd/(1+\beta)}) \geq d^{-s/(1+\beta/2)}
\end{aligned}$$

In particular, notice that for sufficiently large d and fixed β we have that $(1 - d^{-1})^d \geq d^{-(1/2 - 1/(2(1+\beta/2)))}$. This follows from the fact that the left hand side is an increasing function whose limit is $1/e$ and the right hand side is a decreasing function (with β fixed) whose limit is 0. Using this, we get that $(1-d^{-1})^{ds} \geq d^{-s(1/2-1/(2(1+\beta/2)))}$. Moreover, since δ is fixed by β, we have that for large enough values of d, $1-d^{-\delta^2/(1-\delta)^2} > 1/2$. Therefore,

$$\Pr(x' \text{ satisfies } c) \geq (1-d^{-1})^{ds} d^{-s/(1+\beta/2)} (1/2)^s \geq d^{-s(1/2+1/(2(1+\beta/2))} 2^{-s},$$

which is what we wanted to show. □

We also get that for large d, $d^{1/2+1/(2(1+\beta/2))} \geq 2e$. Therefore, $d^{-k(1/2+1/(2(1+\beta/2))} 2^{-k} \geq e^k/d^k$. Moreover, in the case where $||z_c||^2 \leq 1/(sd)$ we have that the random assignment x'' will satisfy c with probability $1/d^k \geq ||z_c||^2(sd)/d^k$. What remains at this point is to analyze the probability that any clause will be satisfied. Therefore we have that

$$\Pr(\text{c is satisfied}) = \frac{(1-k/e^k)||z_c||^2 kd + (k/e^k)e^k}{d^k}$$

It is worth noting that if $1/d > ||z_c||^2 > 1/(kd)$, the probability that c is satisfied is at least k/d^k. This probability yields an approximation ratio of kd/d^k since $k/d^k \geq (kd/d^k)(1/d)$. In the other case, the approximation ratio is given by $c_k kd/d^k$ where $\lim_{k\to\infty} c_k = 1$.

□

4 Rounding for Non-Uniform Solutions

We will now show how to round solutions that are not uniform. In this case, we will also distinguish between two different approaches, which exploit the number of 'short' vectors in a solution in order to form a reduction to the uniform case or utilize a different scheme altogether.

Definition. Let S_u be the indices of the shortest αd vectors among u_i. Let L_u be the complement of S_u with respect to $[d]$, so that $|L_u| = (1-\alpha)d$.

Definition. For a clause c, let $r(C) = |\{(u,i) \in C | i \in S_u\}|$.

Claim 2. *For every $i \in S_u$, $||u_i||^2 \leq 1/((1-\alpha)d)$. Moreover, $\sum_{i\in S_u} ||u_i||^2 \leq \alpha$ and $\sum_{i \in L_u} ||u_i||^2 \geq (1-\alpha)$.*

Proof. By the SDP constraint we get that $||u_i||^2 + \sum_{j\in L_u} ||u_j||^2 \leq 1$. We know that for all $j \in L_u$, $||u_i||^2 \leq ||u_j||^2$. Therefore, $||u_i||^2 \leq 1/(|L_u|+1) \leq 1/((1-\alpha)d)$.

It is clear that proving either part of the second statement directly implies the other. In particular, we have that $\sum_{i \in L_u} ||u_i||^2/(1-\alpha) \geq \sum_{i \in S_u} ||u_i||^2/\alpha$. Shifting the constants to the right hand side and adding $\sum_{i \in S_u} ||u_i||^2$ to both sides yields $1 \geq \frac{(1-\alpha)}{\alpha} \sum_{i \in S_u} ||u_i||^2 + \sum_{i \in S_u} ||u_i||^2 = (1/\alpha) \sum_{i \in S_u} ||u_i||^2$. The claim follows directly from this.

□

For now, let $\alpha = 1/2 + 1/(2d)$ so that $|S_u| = \lceil d/2 \rceil$, $|L_u| = \lfloor d/2 \rfloor$.

Lemma 4. *There exists a randomized polynomial-time algorithm that given an instance I of the Max $k-$CSP(d) problem and an SDP solution, outputs an assignment x such that for every clause $c \in C$*

$$\Pr(x \textit{ satisfies } C) = \frac{b_k c_k ||z_c||^2 (|c|d/8)}{d^{|c|}}$$

where $\lim_{k \to \infty} b_k c_k = 1/8$.

Proof. We will use two different rounding schemes; one will be more efficient when $r(C) \leq 1/4$ and the other when $r(C) \geq 1/4$. We will then weight each scheme appropriately in order to guarantee the result claimed above.

Lemma 5. *There is a polynomial-time randomized rounding algorithms that given a Max $k-CSP(d)$ instance with large d outputs an assignment x such that every clause in x where $r(C) \geq 1/4$ is satisfied with probability at least*

$$\Pr(c \textit{ is satisfied}) = c_k ||z_c||^2 (kd/8)/d^k \tag{1}$$

where c_k is a constant such that $\lim_{k \to \infty} c_k = 1$.

Proof. We will construct a sub instance I' with uniform SDP solution and use the rounding scheme from the previous section to solve it. We first construct a partial assignment x. For $u \in X$, with probability $(1-\alpha)$ assign it a uniform random index from L_u and with probability α leave it unassigned. Let $A = \{u : x_u \text{ is assigned}\}$. We say that a clause c survives the partial assignment x if for every $(u,i) \in c$, either $u \in A$ and $x_u = i$ or $u \notin A$ and $i \in S_u$.

The probability that a clause c survives is

$$\Pr(\text{c survives}) = \prod_{(u,i) \in c, i \in L_u} \Pr(x_u \text{ is assigned } i) \prod_{(u,i) \in c, i \in S_u} \Pr(x_u \text{ is unassigned})$$

$$\left((1-\alpha) \cdot \frac{1}{(1-\alpha)d}\right)^{|c|-r(c)} \cdot \alpha^{r(c)} = \frac{\alpha^{r(c)}}{d^{|c|-r(c)}}$$

For every surviving clause, let $c' = \{(u,i) : u \notin A\}$. Note that if a clause has survived, then for all $(u,i) \in c'$, $i \in S_u$. We get a sub instance of the problem on the set of unassigned variables $\{x_u : u \notin A\}$ with the set of clauses $\{c' : c' \in C \text{ survives}\}$ whose length is $r(c)$. Since all $i \in S_u$ we get a domain of size $|S_u| = \alpha d$.

We now transform the SDP solution of the original instance I into an SDP solution for the new instance I'. We let $u'_i = u_i$ and $z'_c = z_c$. We can remove vectors of non- surviving clauses, and remove vectors of variables that have already been assigned. This SDP solution is a uniform solution on the new instance since $||u_i||^2 \leq 1/((1-\alpha)d) \leq 1/(\alpha d)$, since $\alpha \leq 1/2 + 1/2d$. We run the scheme from the previous section, which will assign values to the unassigned variables. For surviving clauses we get

$$\begin{aligned}\Pr(c' \text{ is satisfied by x}) \geq & c_k ||z_c||^2 (d' r(c))/(d')^{r(c)} \\ & \geq c_k/8 ||z_c||^2 \alpha d |C|/(\alpha d)^{r(c)}\end{aligned}$$

Thus, for every clause c

$$\begin{aligned}\Pr(c \text{ is satisfied by } x) \geq & \Pr(c' \text{ is satisfied by } x) \Pr(c \text{ survives}) \\ & \geq c_k ||z_c||^2 (d' r(c))/(d')^{r(c)} \cdot \alpha^{r(c)}/d^{|c|-r(c)} \\ & \geq (c_k/8) ||z_c||^2 \alpha d |C|/(\alpha d)^{r(c)} \cdot \alpha^{r(c)}/d^{|c|-r(c)} \\ & \geq (c_k/8) ||z_c||^2 d |c| \alpha / d^{|c|} \\ & \geq (c_k/8) ||z_c||^2 d |c| / d^{|c|}\end{aligned}$$

and the result follows from this.

□

Now we consider the case where $r(c) \leq 1/4$.

Lemma 6. *There is a polynomial-time randomized rounding algorithm that given an instance I of the Max $k-$CSP(d) problem outputs an assignment x such that every clause C with $r(c) \leq |c|/4$ is satisfied with probability at least* $\min\{e^{|c|/8}, d||z_c||^2(1+\beta)^{|c|-1}\}/2d^{|c|}$, *where $\beta > 0$ is a constant .*

Proof. We distinguish two different cases and present algorithms for each.

Claim 3. *There is a polynomial-time randomized rounding algorithm that given an instance I of the Max $k-$CSP(d) problem with $||z_c||^2 < (1+\beta)/d$ (for any $\beta > 0$) outputs an assignment x such that every clause C with $r(c) \leq |c|/4$ is satisfied with probability at least* $e^{|c|/8}/d^{|c|}$.

Proof. For each $u \in X$ do the following independently: with probability $3/4$ choose x_u uniformly at random from L_u, with probability $1/4$ choose x_u from S_u uniformly at random. The probability a clause is satisfied is given by

$$\begin{aligned}
\Pr\left(c \text{ is satisfied}\right) &= \prod_{(u,i)\in c, i\in L_u} \left(\frac{3}{4L_u}\right) \prod_{(u,i)\in c, i\in S_u} \left(\frac{1}{4S_u}\right) \\
&= 1/d^{|c|} \left(\frac{3d}{4L_u}\right)^{|c|-r(c)} \left(\frac{d}{4|S_u|}\right)^{r(c)} \\
&\geq 1/d^{|c|} \left(\frac{3d}{4L_u}\right)^{3|c|/4} \left(\frac{d}{4|S_u|}\right)^{|c|/4} \\
&\geq 1/d^{|c|} \left(\left(\frac{3}{2}\right)^{3/4} \left(\frac{d}{2(d+1)}\right)\right)^{|c|}
\end{aligned}$$

Notice that $\left(\frac{3}{2}\right)^{3/4}\left(\frac{d}{2(d+1)}\right) \geq e^{1/8}$. Therefore, the probability that this assignment satisfies c is at least $e^{|c|/8}/d^{|c|}$.

□

Claim 4. *There is a polynomial-time randomized rounding algorithm that given an instance I of the Max $k-$CSP(d) problem with $||z_c||^2 \geq (1+\beta)/d$ (for some $\beta > 0$) outputs an assignment x such that every clause C with $r(c) \leq |c|/4$ is satisfied with probability at least $d||z_c||^2(1+\beta)^{|c|-1}/d^{|c|}$.*

Proof. For each $u \in X$ do the following independently: with probability $\rho_{u,i} = ||u_i||^2/(\sum_{i\in[d]}||u_i||^2)$ set $x_u = i$. Notice that from the SDP constraints we have that if $(u,i) \in c$ then $||u_i||^2 \geq ||z_c||^2$. Moreover, since $(\sum_{i\in[d]}||u_i||^2) \leq 1$ we have that $\rho_{u,i} \geq ||u_i||^2$. Therefore, the probability that this assignment satisfies c is

$$\Pr\left(c \text{ is satisfied}\right) = \prod_{(u,i)\in c} \rho_{u,i} \geq \frac{||z_c||^2 d(1+\beta)^{|c|-1}}{d^{|c|}}$$

□

The lemma stated above follows from applying these two claims with equal weights. In particular, notice that these probabilities are greater than $\frac{|c|d||z_c||^2}{d^{|c|}}$. If $||z_c||^2 \leq 1/d$, then the first assignment assignment satisfies the clause with probability at least $e^{|c|/8}/d^{|c|} \geq ||z_c||^2|c|d/d^{|c|}$. Similarly, if $1 >$

$||z_c||^2 > 1/d$, then the second assignment satisfies the clause with probability at least $d||z_c||^2(1+\beta)^{|c|-1}/d^{|c|} \geq |c|d||z_c||^2/d^{|c|}$.

□

We are now in a position to prove Lemma 4. Consider the following algorithm. With probability $b_k = (1 - k/e^{k/8})$ use the assignment provided in Lemma 5 and with probability $k/e^{k/8}$ use the assignment provided in Lemma 6. Therefore, the probability that a clause c is satisfied is at least

$$\Pr(c \text{ is satisfied}) \geq \frac{b_k c_k ||z_c||^2 (|c|d/8)}{d^{|c|}}$$

where the constants b_k, c_k are such that $\lim_{k\to\infty} b_k c_k = 1$.

□

5 Approximation Algorithm for Max k-CSP(d)

Theorem 1. *There is a polynomial-time randomized approximation algorithm for Max k-CSP(d) that given an instance I finds an assignment that satisfies at least* $b_k c_k (kd/8) OPT(I)/d^k$ *clauses with constant probability.*

Proof. Assume for now that d is large and that $kd/d^k \geq 1/|C|$, as otherwise we simply choose one clause from C and satisfy it. We solve the SDP relaxation and run the rounding scheme from Lemma 4.1 d^k and out put the best solutions. By the guarantees above we will get a solution with expected value at least

$$\begin{aligned}\sum_{c\in C} b_k c_k (|c|d/8)||z_c||^2/d^{|c|} &\geq b_k c_k (|c|d/8)\mathrm{SDP}(I)/d^{|c|} \\ &\geq b_k c_k (|c|d/8)\mathrm{OPT}(I)/d^{|c|}\end{aligned}$$

Let $\mu = b_k c_k (|c|d/8)/d^{|c|}$. Let Z be the random variable equal to the number of satisfied clauses. By the statement above, $\mathrm{OPT}(I) \geq \mathbb{E}[Z] \geq \mu\mathrm{OPT}(I)$. Let $p = \Pr(Z \leq \mu\mathrm{OPT}(I)/2)$. Then

$$p(\mu\mathrm{OPT}(I)/2) + (1-p)\mathrm{OPT}(I) \geq \mathbb{E}[Z] \geq \mu\mathrm{OPT}(I)$$

From this we get that $p \leq 1 - \frac{\mu}{2-\mu}$ So with probability at least $1-p \geq \frac{\mu}{2-\mu}$, we will obtain a solution whose value is at least $\mathrm{OPT}(I)/2$ in one iteration. Performing d^k such iterations, we will find a solution of value $\mu\mathrm{OPT}(I)/2$ with constant probability.

□

References

[1] Per Austrin and Elchanan Mossel: Approxmiation resistant predicates from pairwise independence. *Comput. Complexity*, 18(2):249-271, 2009.

[2] Siu On Chan: Approxmiation resistance from pairwise independent subgroups. In *Proc. 45th STOC*, pp. 447-456. ACM Press, 2013.

[3] Venkatesan Guruswami and Prasad Rhagavendra: Constraint satisfaction over a non-Boolean domain: Approximation algorithms and Unige-Games hardness. In *Proc. 11th Internat. Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX'08)*, pp. 77-90. Springer-Verlag, 2008.

[4] Konstantin Makarychev and Yury Makarychev: Approximation Algorithm for Non-Boolean MAX k-CSP. APPROX 2012, pp. 254-265.

[5] Prasad Rhagavendra: Optimal algorithms and inapproximability results for CSP. In *Proc. 40th STOC*, pp. 245-254. ACM Press, 2008.

[6] Luca Trevisan: Parallel approximation algorithms by positive linear programming. *Algorithmica* 21(1): 72-88, 1998.

[7] Zbynek Šidák: Rectangular confidence regions for the means of multivariate normal distributions. *Journal of the American Statistical Association*, 62(318):626-633, June 1967.